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Position observables for relativistic systems

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Abstract. An elementary proof of the results of Wightman concerning the existence and non-existence of position observables for relativistic systems is presented. The method uses an explicitly described unitary transformation to relate the representations of the Euclidean group obtained by restricting the $m \neq 0$ representations of the Poincaré group to representations having position observables. The position observables are displayed explicitly as differential operators in the cases in which they exist.

1. Introduction

Newton and Wigner (1949) clarified the question of the existence of position observables for relativistic particles. They laid down criteria for a state to be localized at a point of three-space at a given time, and used the Bargmann–Wigner equations to describe the particles. They concluded that particles with strictly positive mass have position observables; in the spin-$\frac{1}{2}$ case their operators are identical with those obtained by the Foldy and Wouthuysen (1950) 'mean position operators'. The basic idea behind the description of the particles is Wigner's notion of an elementary system as being a set of states forming a representation space for an irreducible representation of the inhomogeneous Lorentz group; the Bargmann–Wigner equations were constructed to provide a manifold of solutions having just this property. It is not evident how the Newton–Wigner argument can be made rigorous as it stands. Wightman pointed out that the criteria for the localizability of a state at a point should be replaced by criteria for localizability in a region of three-space, and that if this is done the question can be settled by reference to the irreducible representations themselves rather than by going through the intermediate step of the Bargmann–Wigner equations. This he did in a self-contained paper (Wightman 1962) the major part of which was written in 1952. The essential step was the use of Mackey's imprimitivity theorem (Mackey 1949, 1955); independently Mackey realized that his theorem could settle questions of localizability in quantum theory and he summarized his treatment in his Stillwater Colloquium lectures in 1961 (Mackey 1963).

It turns out that an irreducible representation of the inhomogeneous Lorentz group is localizable in Wightman's sense if, and only if, its restriction to the Euclidean group is unitarily equivalent to the representation

$$V(\boldsymbol{a}, A)f(\boldsymbol{x}) = D(A)f(A^{-1}(\boldsymbol{x} - \boldsymbol{a}))$$

on the space of vector-valued square-integrable functions on $\mathscr{R}^3$, where $\boldsymbol{a}$ is a translation, A a rotation, and D is a representation of the rotation group. In this paper we describe briefly this result and show that an elementary system with mass m, strictly positive, is localizable by displaying the unitary equivalence explicitly. The connection with the Newton–Wigner work is then clear, and their position observables are obtained explicitly.

To discuss the zero-mass case we again make use of an explicitly stated unitary equivalence to show that unless the spin is zero the restriction of the representation to the Euclidean group is not equivalent to one of the required form.

2. Localizability

Following Wightman (1962) and Mackey (1963), we say that a representation U of the Poincaré group defines a localizable system if:

(i) there exists a projection-valued measure P on the Borel sets of three-dimensional Euclidean space $\mathscr{R}^3$. To each Borel set M of $\mathscr{R}^3$ is associated a projection P_M on $\mathfrak{h}(U)$

satisfying:

(*a*) $P_{\bigcup M_i} = \sum_i P_{M_i}$ whenever $M_i \cap M_j = \Phi$ for all $i, j, i \neq j$,

(*b*) $P_{M \cap M'} = P_M P_{M'} = P_{M'} P_M$,

(*c*) $P_{\mathscr{R}^3} = I$;

(ii) $P_{\alpha[M]} = U(\alpha) P_M U(\alpha)^{-1}$, for all α belonging to the Euclidean group $\mathscr{E}(3)$, which is a subgroup of the Poincaré group.

Equivalently, U defines a localizable system if, and only if, U restricted to $\mathscr{E}(3)$ has a transitive system of imprimitivity on $\mathscr{R}^3$. Using Mackey's method, all unitary representations of $\mathscr{E}(3)$ possessing such a system of imprimitivity can be determined.

Theorem A. A unitary representation U of $\mathscr{E}(3)$ has a transitive system of imprimitivity P based on three-dimensional Euclidean space if, and only if, it is unitarily equivalent to a representation V on $L^2(\mathscr{R}^3, \mathfrak{h}, d^3\boldsymbol{x})$ (square-integrable functions on $\mathscr{R}^3$ with values in $\mathfrak{h}$), defined by

$$V(\boldsymbol{a}, A) f(\boldsymbol{x}) = \mathscr{D}(A) f(A^{-1}(\boldsymbol{x} - \boldsymbol{a}))$$

where $\mathscr{D}$ is a unitary representation of SO(3) on $\mathfrak{h}$. The system of imprimitivity associated with V is given by $(P_M f)(\boldsymbol{x}) = \chi_M(\boldsymbol{x}) f(\boldsymbol{x})$ where χ_M is the characteristic function of the Borel set M.

This result may be found in Wightman (1962) and Mackey (1967).

3. The representation $[m, s]$ of the Poincaré group, for $m > 0$

This representation takes the form

$$U(a, \Lambda) f(p) = \exp \mathrm{i}\{p, a\} Q(p, \Lambda) f(\Lambda^{-1} p)$$

where $\{,\}$ is the Lorentz bracket. $Q(p, \Lambda)$ may be expressed as $\mathscr{D}^s(\Lambda_p \Lambda \Lambda_{\Lambda_p^{-1}}{}^{-1})$ where $\mathscr{D}^s$ is the usual $(2s+1)$-dimensional irreducible representation of SU(2, C) on $\mathfrak{h}_s$. Λ_p is a coset representative of the cosets of SU(2, C) in SL(2, C) and satisfies $\Lambda_p p = k$, $k = (m, 0, 0, 0)$. p satisfies $\{p, p\} = m^2$ and f is a $(2s+1)$-component function on the positive mass hyperboloid $p^2 = m^2$, H_m, square integrable with respect to the measure $d^3\boldsymbol{p}/p^0 = d\mu$. Thus U is a representation on $L^2(H_m, \mathfrak{h}_s, d\mu)$ (Moussa and Stora 1964).

Restricting ourselves to the Euclidean group $\mathscr{E}(3)$ we have, putting $A^{-1}p = q$,

$$U(\boldsymbol{a}, A) f(p) = \exp(-\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{a}) \mathscr{D}^s(\Lambda_p A \Lambda_q{}^{-1}) f(q).$$

Now A leaves p^0 invariant. We may write $\Lambda_p = \Lambda_\theta A_p$ where $A_p p = (m \cosh\theta, 0, 0, m \sinh\theta)$ and $m \cosh\theta = p^0$. $\Lambda_\theta(m \cosh\theta, 0, 0, m \sinh\theta) = (m, 0, 0, 0)$, Λ_θ being implemented by means of the matrix

$$\begin{pmatrix} e^{-\theta/2} & 0 \\ 0 & e^{\theta/2} \end{pmatrix}$$

of SL(2, C). We choose A_p to belong to SU(2, C). Thus $A_p A A_q{}^{-1}$ belongs to SU(2, C). Now $(A_p A A_q{}^{-1})(m \cosh\theta, 0, 0, m \sinh\theta) = (m \cosh\theta, 0, 0, m \sinh\theta)$ for all m and θ. Hence $A_p A A_q{}^{-1}$ is of the form

$$\begin{pmatrix} e^{\mathrm{i}\phi/2} & 0 \\ 0 & e^{-\mathrm{i}\phi/2} \end{pmatrix}$$

and thus commutes with Λ_θ. We have

$$\Lambda_p A \Lambda_q{}^{-1} = \Lambda_\theta A_p A A_q{}^{-1} \Lambda_\theta{}^{-1} = A_p A A_q{}^{-1}.$$

Hence this becomes: $U(\boldsymbol{a}, A) f(p) = \exp(-\mathrm{i}\boldsymbol{p} . \boldsymbol{a}) \mathscr{D}^s(A_p A A_q{}^{-1}) f(q)$. Now consider the mapping $f(p) \to \mathscr{D}^s(A_p^{-1})f(p) = \phi(\boldsymbol{p})$ where $\boldsymbol{p}^2 = p^{02}-m^2$. This is a unitary map.

$$\mathscr{D}^s(A_p^{-1})U(\boldsymbol{a}, A)\mathscr{D}^s(A_p)\{\mathscr{D}^s(A_p^{-1})f(p)\}$$
$$= \exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{a})\mathscr{D}^s(A_p^{-1})\mathscr{D}^s(A_p)\mathscr{D}^s(A)\mathscr{D}^s(A_q^{-1})f(q).$$

If we write $\tilde{U}(\boldsymbol{a}, A) = \mathscr{D}^s(A_p^{-1})U(\boldsymbol{a}, A)\mathscr{D}^s(A_p)$ we see that U restricted to $\mathscr{E}(3)$ is unitarily equivalent to the representation $\tilde{U}$ of $\mathscr{E}(3)$ on $L^2(H_m, \mathfrak{h}_s, \mathrm{d}\mu)$ given by

$$\tilde{U}(\boldsymbol{a}, A)\phi(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{a})\mathscr{D}^s(A)\phi(A^{-1}\boldsymbol{p})$$

$\boldsymbol{p}^2$ having any positive value, and the measure being

$$\frac{\mathrm{d}^3\boldsymbol{p}}{(m^2+\boldsymbol{p}^2)^{1/2}} = \frac{\mathrm{d}^3\boldsymbol{p}}{p^0} = \mathrm{d}\mu.$$

Consider the operator W such that

$$(W\phi)(\boldsymbol{x}) = (2\pi)^{-3/2}\int \phi(\boldsymbol{p})\exp(\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{x})p^{01/2}\,\mathrm{d}^3\boldsymbol{p}/p^0.$$

W is a unitary operator (see appendix 1) from $L^2(H_m, \mathfrak{h}_s, \mathrm{d}\mu)$ to $L^2(\mathscr{R}^3, \mathfrak{h}_s, \mathrm{d}^3\boldsymbol{x})$. Now define $\tilde{U}'(\boldsymbol{a}, A)(W\phi)(\boldsymbol{x}) = (W\tilde{U}(\boldsymbol{a}, A)\phi)(\boldsymbol{x})$ for all ϕ. Then $\tilde{U}'$ is unitarily equivalent to the representation $\tilde{U}$ of $\mathscr{E}(3)$ and we easily see that $\tilde{U}'(\boldsymbol{a}, A)f(\boldsymbol{x}) = \mathscr{D}^s(A)f(A^{-1}(\boldsymbol{x}-\boldsymbol{a}))$. Hence by theorem A of § 2 $\tilde{U}'$ represents a localizable system and it follows that U also represents a localizable system.

For $\tilde{U}'$ we have that the position observables are represented by multiplication operators x_k:

$$x_k(W\phi)(\boldsymbol{x}) = (W(Q_k)_{\mathrm{op}}\phi)(\boldsymbol{x}).$$

Therefore

$$(Q_k)_{\mathrm{op}}\phi(\boldsymbol{p}) = (W^{-1}x_k W\phi)(\boldsymbol{p})$$
$$= \frac{1}{(2\pi)^3}\int \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{x}}x_k p^{01/2}\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{e}^{\mathrm{i}\boldsymbol{x}\,.\,\boldsymbol{p}'}\phi(\boldsymbol{p}')\frac{\mathrm{d}^3\boldsymbol{p}'}{p'^{01/2}}.$$

Therefore

$$(Q_k)_{\mathrm{op}}\phi(\boldsymbol{p}) = \mathrm{i}\left(\frac{\partial}{\partial p^k} - \frac{p^k}{2p^{02}}\right)\phi(\boldsymbol{p})$$

when $\phi(\boldsymbol{p})$ is a function in the Hilbert space of the representation $\tilde{U}$ of $\mathscr{E}(3)$.

4. The representation [0, *s*] of the Poincaré group where *s* has any half-integral value

The representation $[0, s]$ takes the form $U(a, \Lambda)f(p) = \exp(\mathrm{i}\{p, a\})L(p, \Lambda)f(\Lambda^{-1}p)$ where $p^2 = 0$. $L(p, \Lambda)$ may be written as $L^s(\Lambda_p\Lambda\Lambda_{\Lambda_p^{-1}}{}^{-1})$ where L^s is a representation of the universal covering group of $\mathscr{E}(2)$:

$$L^s : A(\phi, z) = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\phi/2} & z\,\mathrm{e}^{-\mathrm{i}\phi/2} \\ 0 & \mathrm{e}^{-\mathrm{i}\phi/2} \end{pmatrix} \to \exp(\mathrm{i}s\phi).$$

Each Λ_p is the representative of a coset of $\mathscr{E}(2)$ in SL(2, C) and satisfies $\Lambda_p p = n = (1, 0, 0, 1)$.

Let $(\boldsymbol{a}, A)$ belong to $\mathscr{E}(3)$. Then we have, putting $A^{-1}p = q$,

$$U(\boldsymbol{a}, A)f(p) = \exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{a})L^s(\Lambda_p A\Lambda_q^{-1})f(q).$$

We may write $\Lambda_p = \Lambda_{p^0}A_p$ where $A_p p = (p^0, 0, 0, p^0) = p^0 n$ and $\Lambda_{p^0}(p^0 n) = n$,

$$\Lambda_{p^0} = \begin{pmatrix} p^{0-1/2} & 0 \\ 0 & p^{01/2} \end{pmatrix}.$$

Then we may choose A_p belonging to SU(2, C). In this case $A_pAA_q{}^{-1}$ has the form

$$\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$$

for some ϕ and we have

$$L^s(\Lambda_p A \Lambda_q{}^{-1}) = L^s(\Lambda_{p^0} A_p A A_q{}^{-1} \Lambda_{p^0}{}^{-1}) = L^s(A_p A A_q{}^{-1}).$$

Thus we may write

$$U(\boldsymbol{a}, A)f(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{a})L^s(A_pAA_q{}^{-1})f(\boldsymbol{q})$$

where $\boldsymbol{p}^2 = p^{02}$, thus taking any positive value, and L^s is the representation

$$A_\phi = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \to \exp(\mathrm{i}s\phi)$$

of SO(2). It is now clear that this is the representation $\int U^{s,m}\,\mathrm{d}m$ of $\mathscr{E}(3)$, where $U^{s,m}$ is an irreducible representation of $\mathscr{E}(3)$ on the sphere $m^2 = p^2$. $U^{s,m}$ can be expressed as follows:

$$U^{s,m}(\boldsymbol{a}, A)f(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{a})L^s(A_pAA_q{}^{-1})f(\boldsymbol{q}) \qquad \text{where } \boldsymbol{p}^2 = m^2$$

$$A_p\boldsymbol{p} = (0, 0, m) \quad \text{and} \quad L^s(A_\phi) = \exp(\mathrm{i}s\phi).$$

Now, consider the representation $V^{s,m} \simeq U^{s,m} \oplus U^{s-1,m} \oplus \ldots \oplus U^{-s,m}$ of $\mathscr{E}(3)$. We have

$$V^{s,m}(\boldsymbol{a}, A)\boldsymbol{f}(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{a})\Lambda^s(\boldsymbol{p}, A)\boldsymbol{f}(\boldsymbol{q})$$

where $\Lambda^s(\boldsymbol{p}, A)$ is a diagonal $(2s+1)\times(2s+1)$ matrix with $L^j(A_pAA_q{}^{-1})$, $j = s, s-1, \ldots, -s$, as its diagonal entries. It follows that if $A_pAA_q{}^{-1} = A_\phi$, $\Lambda^s(\boldsymbol{p}, A)$ has diagonal entries $\exp(ij\phi)$. Therefore, we know that $\Lambda^s(\boldsymbol{p}, A) = \mathscr{D}^s(A_pAA_q{}^{-1})$ where $\mathscr{D}^s$ is the $(2s+1)$-dimensional irreducible representation of SU(2, C). Thus

$$V^{s,m}(\boldsymbol{a}, A)\boldsymbol{f}(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{a})\mathscr{D}^s(A_p)\mathscr{D}^s(A)\mathscr{D}^s(A_q{}^{-1})\boldsymbol{f}(\boldsymbol{q}).$$

Now map $\boldsymbol{f}(\boldsymbol{p}) \to \mathscr{D}^s(A_p{}^{-1})\boldsymbol{f}(\boldsymbol{p})$ and we see that $V^{s,m}$ is unitarily equivalent to the representation $\tilde{V}^{s,m}(\boldsymbol{a}, A)\phi(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{a})\mathscr{D}^s(A)\phi(A^{-1}\boldsymbol{p})$ where $\boldsymbol{p}^2 = m^2$. We have shown that the representation

$$U^s(\boldsymbol{a}, A)\phi(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p}\,.\,\boldsymbol{a})\mathscr{D}^s(A)\phi(A^{-1}\boldsymbol{p})$$

where $\boldsymbol{p}^2$ has any value, is unitarily equivalent to the representation

$$\int \sum_{j=-s}^{+s} \oplus\, U^{j,m}\,\mathrm{d}m$$

where $U^{j,m}$ is the irreducible representation of $\mathscr{E}(3)$ defined above.

From § 3 we know that any representation of the form $V(\boldsymbol{a}, A)f(\boldsymbol{x}) = \mathscr{D}(A)f(A^{-1}(\boldsymbol{x}-\boldsymbol{a}))$ is equivalent to a representation of the form $U(\boldsymbol{a}, A)\phi(\boldsymbol{p}) = \exp(-\mathrm{i}\boldsymbol{p}.\boldsymbol{a})\mathscr{D}(A)\phi(A^{-1}\boldsymbol{p})$ and, since $\mathscr{D}$ is a representation of SU(2, C), it follows that $U \simeq \Sigma_s \alpha_s U^s$ where α_s are positive integers.

$[0, s]$ restricted to $\mathscr{E}(3)$ is equivalent to $\int U^{s,m}\,\mathrm{d}m$. From the above, it is clear that this representation is equivalent to a proper sub-representation of the representation U^i for all $i \geqslant |s|$ if $s \neq 0$. $[0, 0]$ restricted to $\mathscr{E}(3)$ is equivalent to U^0. Thus by theorem A $[0, s]$ represents a localizable system if, and only if, $s = 0$.

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Appendix

We prove that W, as defined in § 3, is a unitary operator. For simplicity consider the case when $s = 0$ and $\phi \in L^2(H_m, \mathfrak{h}^0, d\mu)$ has a single component. The following extends easily to functions with $(2s+1)$ components.

$$W\phi(\boldsymbol{x}) = (2\pi)^{-3/2} \int \exp(i\boldsymbol{p}\,.\,\boldsymbol{x}) p^{0\,1/2}\phi(\boldsymbol{p}) \frac{d^3\boldsymbol{p}}{p^0}.$$

The inverse transformation is given by

$$(W^{-1}f)(\boldsymbol{p}) = (2\pi)^{-3/2} \int \exp(-i\boldsymbol{p}\,.\,\boldsymbol{x}) p^{0\,1/2} f(\boldsymbol{x})\, d^3\boldsymbol{x}.$$

Firstly W is an isometry. Let F be the three-dimensional Fourier transform. Then

$$\begin{aligned} \|W\phi\|^2 &= \int |W\phi|^2\, d^3\boldsymbol{x} \\ &= \int |F(p^{0\,-1/2}\phi)|^2\, d^3\boldsymbol{x} \\ &= \int |p^{0\,-1/2}\phi|^2\, d^3\boldsymbol{p} \\ &= \int |\phi|^2 \frac{d^3\boldsymbol{p}}{p^0} = \|\phi\|^2. \end{aligned}$$

The functions $\{p^{0\,1/2}h_l(p^1)h_m(p^2)h_n(p^3)\}_{l,m,n}$ form a basis for $L^2(H_m, \mathfrak{h}^0, d\mu)$, where h_i is the ith Hermite function of a single variable.

$$\begin{aligned} (Wp^{0\,1/2}h_l h_m h_n)(\boldsymbol{x}) &= (2\pi)^{-3/2} \int \exp(i\boldsymbol{p}\,.\,\boldsymbol{x}) p^{0\,1/2} p^{0\,1/2} h_l h_m h_n \frac{d^3\boldsymbol{p}}{p^0} \\ &= i^{l+m+n} h_l(x_1) h_m(x_2) h_n(x_3). \end{aligned}$$

Hence W has a dense range and thus is a unitary operator mapping $L^2(H_m, \mathfrak{h}^0, d\mu)$ to $L^2(\mathscr{R}^3, \mathfrak{h}^0, d^3\boldsymbol{x})$.

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